Choquet optimal solutions: where are they?

Thibaut Lust
Université Pierre et Marie Curie (LIP6)
Antoine Rolland
Université de Lyon 2 (ERIC)

Bruxelles - 9 juin 2017

Plan

Our motivation in two questions

Framework : optimisation problem

⇒ what is the benefit to use Choquet integral instead of weighted sum?

Question 1: in the space of solutions to an optimisation problem, where are solutions which are Choquet-optimal without being weighted-sum optimal?

Question 2: in the space of capacities, which capacities give solutions which are Choquet optimal without being weighted-sum optimal?

Joint work with Thibaut Lust 

- T. Lust (2015), *Choquet integral versus weighted sum in multicriteria decision contexts*, In: Walsh T. (eds) Algorithmic Decision Theory. ADT 2015. Lecture Notes in Computer Science, vol 9346. Springer, Cham.
- T. Lust and A. Rolland (2014), *"2-additive Choquet Optimal Solutions in Multiobjective Optimization Problems"*. In A. Laurent et al. (Eds.): IPMU 2014, Part I, CCIS 442, pp. 256-265, , Montpellier.
- T. Lust and A. Rolland (2013), *"On the computation of Choquet optimal solutions in multicriteria decision contexts"*. In Proceedings of the 7th Multi-disciplinary International Workshop on Artificial Intelligence , S. Ramanna, C. Sombattheera, P. Lingras and A. Krishna (Edts), Krabi, Thailand.
- T. Lust and A. Rolland (2013), *"Choquet optimal set in biobjective combinatorial optimization"*. In Computers and Operations Research , Volume 40, Issue 10, Pages 2260-2269.

Choquet integral

Aggregation operator based on non-additive measures (capacities) allowing to take into account interactions between criteria.

Choquet integral

Aggregation operator based on non-additive measures (capacities) allowing to take into account interactions between criteria.

Capacity

A capacity is a set function v: $2^{\mathcal{P}} \rightarrow [0,1]$ such that:

- $v(\emptyset) = 0, \ v(\mathcal{P}) = 1$ (normality)
- $\forall \mathcal{A}, \mathcal{B} \in 2^{\mathcal{P}} | \mathcal{A} \subseteq \mathcal{B}, v(\mathcal{A}) \leq v(\mathcal{B})$ (monotonicity)

For example, for $p = 3$, we have to set
$v(\{1\}), v(\{2\}), v(\{3\}), v(\{1,2\}), v(\{1,3\})$ and $v(\{2,3\}$.
$(v(\emptyset) = 0$ and $v\{1,2,3\} = 1)$

Choquet integral

Choquet integral

Choquet integral of a vector $y \in \mathbb{Z}^p$ for a capacity v is:

$$f_v^C(y) = \sum_{i=1}^{p} (y_{[i]} - y_{[i-1]}) v(Y_{[i]})$$

where [.] represents the permutation over $\{1, \ldots, p\}$ such that $0 = y_{[0]} \leq y_{[1]} \leq \ldots \leq y_{[p]}$, $Y_{[i]} = \{j \in \{1, \ldots, p\}, y_j \geq y_{[i]}\} = \{[i], [i+1], \ldots, [p]\}$ for $i \leq p$ and $Y_{[p+1]} = \emptyset$.

Example

- $y = (17, 8, 14)$, $y_{[]} = (0, 8, 14, 17)$

  $f_v^C(y) = (8 - 0) * v(\{1, 2, 3\}) + (14 - 8) * v(\{1, 3\}) + (17 - 14) * v(\{1\})$

Choquet Integral

According to the values given to the capacity, the Choquet integal allows to model:

- Min
- Max
- Weighted average
- Ordered weighted average (OWA)
- Weighted ordered weighted average (WOWA)
- Etc.

Choquet Integral

According to the values given to the capacity, the Choquet integal allows to model:

- Min
- Max
- Weighted average
- Ordered weighted average (OWA)
- Weighted ordered weighted average (WOWA)
- Etc.

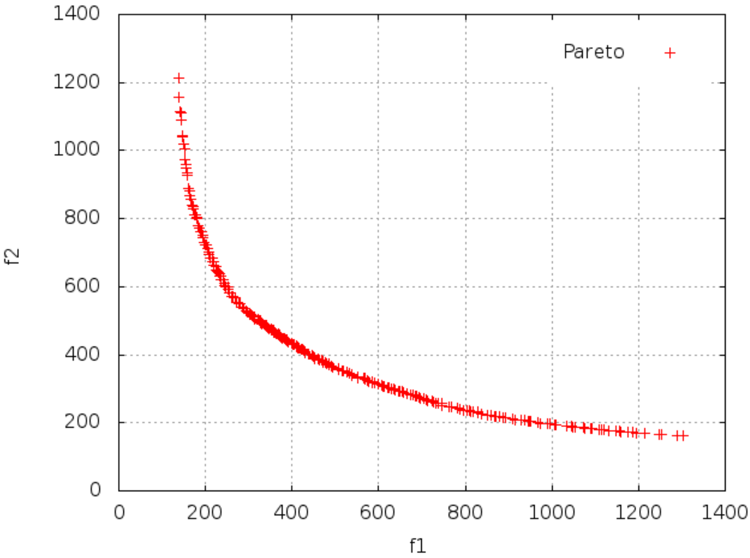
⇒ But does not allow to reach all Pareto optimal solutions!

Optimal solutions of a multicriteria combinatorial approach

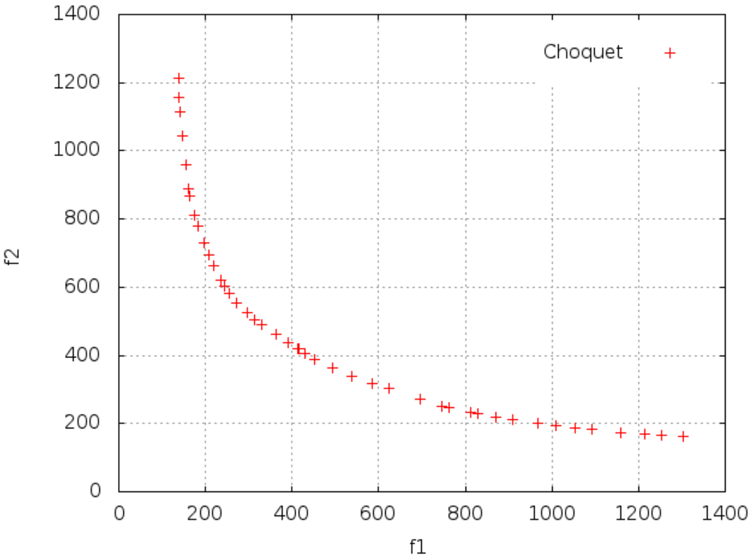
- Pareto front
- Weighted-sum optimal solutions WS-front
- Choquet Optimal Solutions : C-front

→ Generation of the Choquet optimal set: set that contains at least one feasible solution optimal for each possible Choquet integral

Pareto front



Choquet front



Plan

Motivation

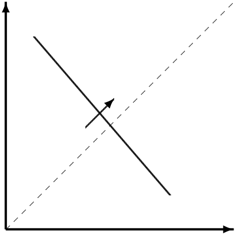
Generation of the Choquet optimal set

Results
- Biobjective combinatorial optimization problems
- Multiobjective combinatorial optimization problems ($p > 2$)

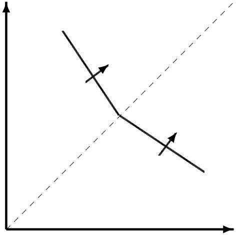
k-additivity

iso-optimal lines

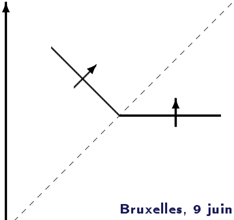
Weighted Sum



OWA

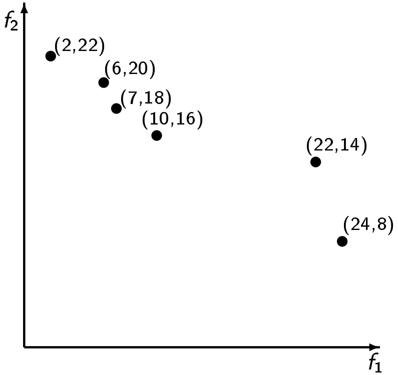


Choquet Integral



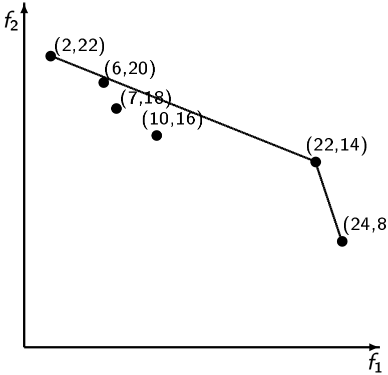
Example with two objectives

- 6 solutions
- Evaluations=$\{(2,22),(6,20),(7,18),(10,16),(22,14),(24,8)\}$



Example with two objectives

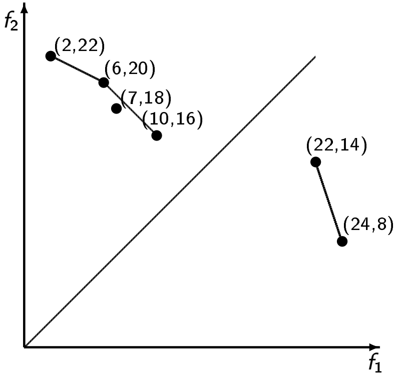
- 6 solutions
- Evaluations=$\{(2,22),(6,20),(7,18),(10,16),(22,14),(24,8)\}$



- (2,22), (22,14) and (24,8) are Choquet optimal because they are supported

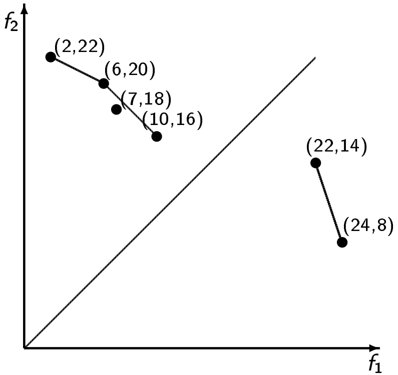
Example with two objectives

- 6 solutions
- Evaluations=$\{(2,22),(6,20),(7,18),(10,16),(22,14),(24,8)\}$



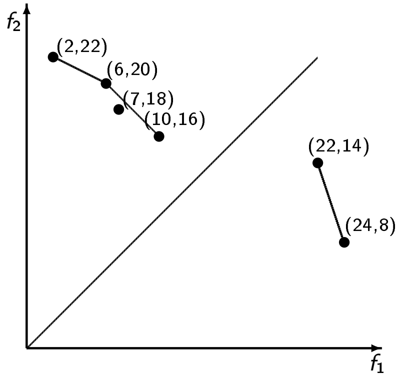
- (2,22), (22,14) and (24,8) are Choquet optimal because they are supported
- (7,18) cannot be Choquet optimal:
  - $f_v^C(6,20) = 6(1-v_2) + 20v_2$
  - $f_v^C(7,18) = 7(1-v_2) + 18v_2$
  - $f_v^C(10,16) = 10(1-v_2) + 16v_2$

Example with two objectives



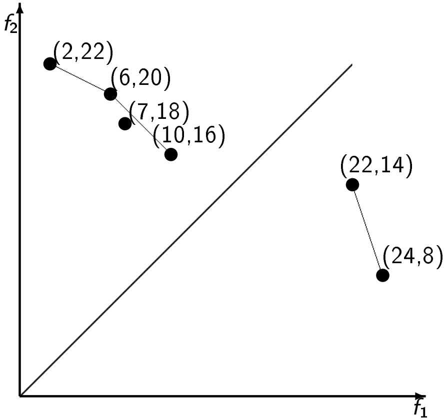
- Necessary condition: a solution is Choquet optimal $\Rightarrow$ the solution optimizes a WS in the subspace ($f_1(x) \leq f_2(x)$ or $f_1(x) \geq f_1(x)$) in which she is located

Example with two objectives

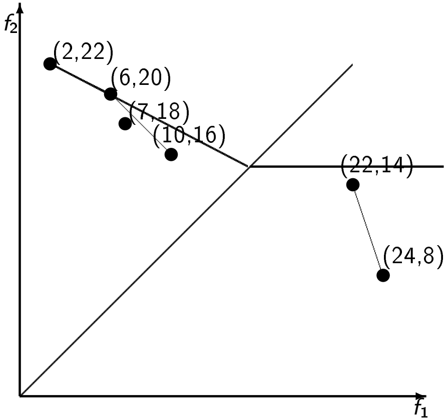


- Necessary condition: a solution is Choquet optimal $\Rightarrow$ the solution optimizes a WS in the subspace ($f_1(x) \leq f_2(x)$ or $f_1(x) \geq f_1(x)$) in which she is located
- Sufficient? No, we also have to check that the solutions located in the other subspaces are not better

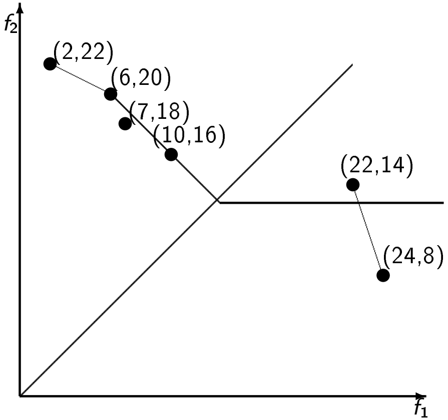
Example with two objectives



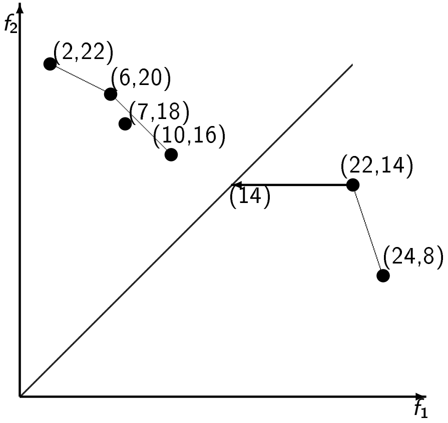
Example with two objectives



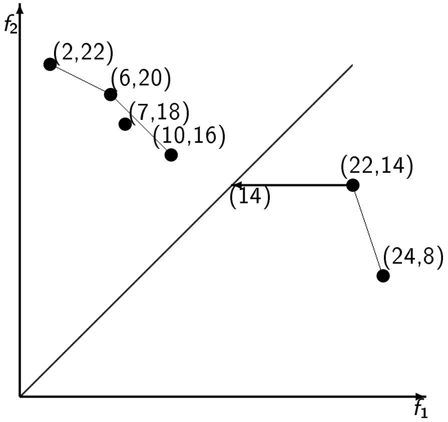
Example with two objectives



Example with two objectives



Example with two objectives



$\Rightarrow \max_{x \in \mathcal{X}} \min(f_1(x), f_2(x))$ plays an important role.

Generation of the Choquet optimal set (two objectives)

Parameters ↓: a MOCO problem
Parameters ↑: the Choquet optimal set $\mathcal{X}_C$

1. Determination of the point $m = (m_1, m_2)$ with $m_1 = m_2 = \max_{x \in \mathcal{X}} \min(f_1(x), f_2(x))$
2. Solve the WS problem $\max\limits_{f_1(x) \geq f_2(x)} f_\lambda^{ws}(f(x))$ with the additional point m
3. Solve the WS problem $\max\limits_{f_2(x) \geq f_1(x)} f_\lambda^{ws}(f(x))$ with the additional point m

Characterization theorem for p objectives

- Let σ be a permutation on $\mathcal{P}$.
- Let O_σ be the subset of points $y \in \mathbb{R}^p$ such that $y \in O_\sigma \iff y_{\sigma_1} \geq y_{\sigma_2} \geq \ldots \geq y_{\sigma_p}$.

Characterization theorem for p objectives

- Let σ be a permutation on $\mathcal{P}$.
- Let O_σ be the subset of points $y \in \mathbb{R}^p$ such that $y \in O_\sigma \iff y_{\sigma_1} \geq y_{\sigma_2} \geq \ldots \geq y_{\sigma_p}$.

Characterization of $\mathcal{Y}_C$ (Choquet optimal set in the objective space)

$$\mathcal{Y}_C \cap O_\sigma = \mathcal{Y} \cap WS(\mathcal{P}_{O_\sigma})$$

- $\mathcal{P}_{O_\sigma}$ is the set containing the projections with the application p_{O_σ}
- $WS(\mathcal{P}_{O_\sigma})$ designs the set of supported points of the set $\mathcal{P}_{O_\sigma}$

Characterization theorem for p objectives

- Let σ be a permutation on $\mathcal{P}$.
- Let O_σ be the subset of points $y \in \mathbb{R}^p$ such that $y \in O_\sigma \iff y_{\sigma_1} \geq y_{\sigma_2} \geq \ldots \geq y_{\sigma_p}$.

Characterization of $\mathcal{Y}_C$ (Choquet optimal set in the objective space)

$$\mathcal{Y}_C \cap O_\sigma = \mathcal{Y} \cap WS(\mathcal{P}_{O_\sigma})$$

- $\mathcal{P}_{O_\sigma}$ is the set containing the projections with the application p_{O_σ}
- $WS(\mathcal{P}_{O_\sigma})$ designs the set of supported points of the set $\mathcal{P}_{O_\sigma}$
- p_{O_σ}: application from $\mathbb{R}^p$ to $\mathbb{R}^p$ such that for all $y \in \mathcal{Y}$:

$$(p_{O_\sigma}(y))_{\sigma_i} = (\min(y_{\sigma_1}, \ldots, y_{\sigma_i})), \forall i \in \mathcal{P}$$

- For example, if $p = 3$, for the permutation (1,2,3), we have:

$$p_{O_\sigma}(y) = \big(\min(y_1), \min(y_1, y_2), \min(y_1, y_2, y_3)\big)$$

Characterization theorem for p objectives

- Let σ be a permutation on $\mathcal{P}$.
- Let O_σ be the subset of points $y \in \mathbb{R}^p$ such that $y \in O_\sigma \iff y_{\sigma_1} \geq y_{\sigma_2} \geq \ldots \geq y_{\sigma_p}$.

Characterization of $\mathcal{Y}_C$ (Choquet optimal set in the objective space)

$$\mathcal{Y}_C \cap O_\sigma = \mathcal{Y} \cap WS(\mathcal{P}_{O_\sigma})$$

- $\mathcal{P}_{O_\sigma}$ is the set containing the projections with the application p_{O_σ}
- $WS(\mathcal{P}_{O_\sigma})$ designs the set of supported points of the set $\mathcal{P}_{O_\sigma}$
- p_{O_σ}: application from $\mathbb{R}^p$ to $\mathbb{R}^p$ such that for all $y \in \mathcal{Y}$:

$$(p_{O_\sigma}(y))_{\sigma_i} = (\min(y_{\sigma_1}, \ldots, y_{\sigma_i})), \forall i \in \mathcal{P}$$

- For example, if $p = 3$, for the permutation (1,2,3), we have:

$$p_{O_\sigma}(y) = \big(\min(y_1), \min(y_1, y_2), \min(y_1, y_2, y_3)\big)$$

- Proof: Comes from the Möbius representation of the Choquet integral:

$$f_v^C(y) = \sum_{\mathcal{A} \subseteq \mathcal{P}} m_v(\mathcal{A}) \min_{i \in \mathcal{A}} y_i$$

Example with three objectives

Case of the subspace $(f_1 \geq f_2 \geq f_3)$:

f_1	f_2	f_3
600	**200**	**100**
700	**100**	**100**
1000	**2**	**1**
500	**500**	**200**
200	300	400
100	400	500
300	320	210
310	5	201
1	1000	2
2	1	1000

f_1	f_2	f_3
600	**200**	**100**
700	**100**	**100**
1000	**2**	**1**
500	**500**	**200**
200	200	200
100	100	100
300	300	210
310	5	5
1	1	1
2	1	1

$$(f_1(x), f_2(x), f_3(x)) \rightarrow (f_1(x), \min(f_1(x), f_2(x)), \min(f_1(x), f_2(x), f_3(x))$$

Plan

Motivation

Generation of the Choquet optimal set

Results
- Biobjective combinatorial optimization problems
- Multiobjective combinatorial optimization problems ($p > 2$)

k-additivity

Plan

Motivation

Generation of the Choquet optimal set

Results

- Biobjective combinatorial optimization problems
- Multiobjective combinatorial optimization problems ($p > 2$)

k-additivity

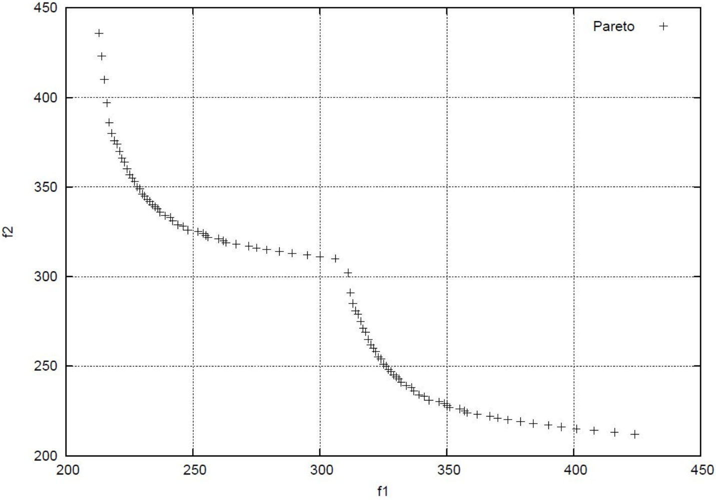
Biobjective knapsack problem

Instances		#			CPU(s)		
Type	n	WS	Choquet	Pareto	WS	Choquet	Pareto
Random	400	68.4	71.3	1713.3	0.007	0.739	307.10
	700	115.0	118.7	4814.8	0.014	1.537	5447.92
Correlated	2000	38.8	39.8	477.7	0.016	0.483	251.06
	4000	76.4	78.4	1542.3	0.046	2.484	6773.26
Uncorrelated	300	66.5	69.9	2893.6	0.011	1.015	373.1
	500	111.9	115.9	7112.1	0.019	9.045	4547.98
Strongly uncorrelated	100	35.5	38.7	1765.4	0.009	1.335	40.87
	200	63.8	68.0	5464.0	0.018	3.328	1145.92

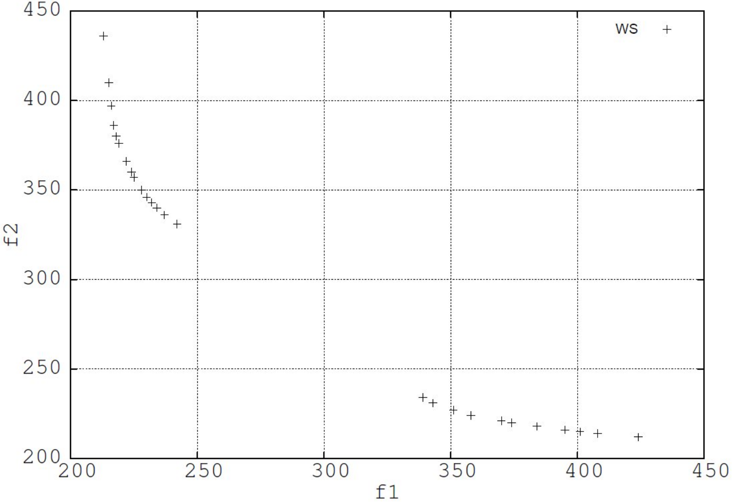
Table: Results for the biobjective knapsack problem.

Biobjective spanning tree

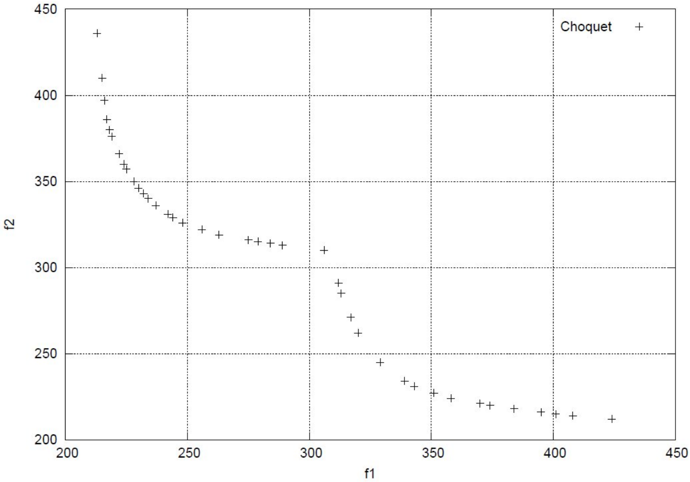
Biobjective spanning tree



Biobjective spanning tree



Biobjective spanning tree



Plan

Motivation

Generation of the Choquet optimal set

Results

- Biobjective combinatorial optimization problems
- Multiobjective combinatorial optimization problems ($p > 2$)

k-additivity

Results for defined Pareto front

Results for defined Pareto front

- We generate 3000 Pareto non dominated solutions of random knapsack instances with a number of criteria included between 2 and 7
- We apply the characterization theorem to determine the Choquet-optimal points

Results for defined Pareto front

- We generate 3000 Pareto non dominated solutions of random knapsack instances with a number of criteria included between 2 and 7
- We apply the characterization theorem to determine the Choquet-optimal points

# Crit	# WS	# C	% C not WS
2	123	128	3.91
3	184	240	23.33
4	240	380	36.84
5	282	485	41.86
6	408	676	39.64
7	528	1016	48.03

Table: Random multiobjective knapsack instances (3000 points).

Plan

k-additive Choquet integral

Möbius transformation

The Möbius transformation of a capacity v is a function $m : 2^{\mathcal{P}} \rightarrow \mathbb{R}$ defined by

$$m(\mathcal{T}) = \sum_{\mathcal{K} \subseteq \mathcal{T}} (-1)^{|\mathcal{T} \setminus \mathcal{K}|} v(\mathcal{K}) \;\; \forall \mathcal{T} \in 2^{\mathcal{P}}$$

k-additive capacity (Grabisch)

A capacity v is said to be k-additive ($k > 0$, $k \leq p$), if its Möbius transformation m satisfies

- $\forall \mathcal{T} \in 2^{\mathcal{P}}, m(\mathcal{T}) = 0$ if $|T| > k$
- $\exists \mathcal{B} \in 2^{\mathcal{P}}$ such that $|B| = k$ and $m(B) \neq 0$.

Example ($p = 4, k = 2$)

$\mathcal{T}$	v	m
$\{1\}$	0.138	0.138
$\{2\}$	0.510	0.510
$\{3\}$	0.351	0.351
$\{4\}$	0	0
$\{1,2\}$	0.510	-0.138
$\{1,3\}$	0.489	0
$\{1,4\}$	0.579	0.441
$\{2,3\}$	0.559	-0.302
$\{2,4\}$	0.510	0
$\{3,4\}$	0.351	0
$\{1,2,3\}$	0.559	0
$\{1,2,4\}$	0.951	0
$\{1,3,4\}$	0.930	0
$\{2,3,4\}$	0.559	0

Advantages: less values have to be fixed ($2^p \leftrightarrow \sum_{l=1}^{p} C_l^p$)

Sufficient condition

- Let σ be a permutation on $\mathcal{P}$.
- Let O_σ be the subset of points $y \in \mathbb{R}^p$ such that
  $y \in O_\sigma \iff y_{\sigma_1} \geq y_{\sigma_2} \geq \ldots \geq y_{\sigma_p}$.

Sufficient condition $\mathcal{Y}_{C^k}$ (k-additive Choquet optimal set in the objective space)

$$\mathcal{Y} \cap WS(\mathcal{P}^k_{O_\sigma}) \Rightarrow \mathcal{Y}_{C^k} \cap O_\sigma$$

- $\mathcal{P}^k_{O_\sigma}$ is the set containing the projections with the application $p^k_{O_\sigma}$
- $WS(\mathcal{P}^k_{O_\sigma})$ designs the set of supported points of the set $\mathcal{P}^k_{O_\sigma}$

$$(p^k_{O_\sigma}(y))_{\sigma_i} = \begin{cases} \min(y_{\sigma_1}, \ldots, y_{\sigma_i}), \forall i \leq k \\ \min(y_{\sigma_1}, \ldots, y_{\sigma_{k-1}}, y_{\sigma_i}), \forall i > k \end{cases}$$

For example, if $p = 4$, for the permutation (1,2,3,4), we have:

- for $k = 2$: $p^2_{O_\sigma}(y) = \big(\min(y_1), \min(y_1, y_2), \min(y_1, y_3), \min(y_1, y_4)\big)$
- for $k = 3$: $p^3_{O_\sigma}(y) = \big(\min(y_1), \min(y_1, y_2), \min(y_1, y_2, y_3), \min(y_1, y_2, y_4)\big)$

k-additivity

k	# k-C (exact)	# k-C (sufficient condition)
1	130	130
2	182	147
3	183	160
4	183	171
5	183	177
6	183	183

Table: Random multiobjective knapsack instances (250 points, 6 objectives).

Thank you for your attention.